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Free convection on a heated vertical plate: the solution for small Prandtl number

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Abstract. The solution of the equations for the free-convection boundary-layer flow on a vertical plate with a prescribed power-law heating is considered for small values of the Prandtl number σ. It is shown that the boundary layer divides up into two regions. There is a thin inner region, of thickness $O(\sigma^{1/10})$, in which, to leading order, the temperature is a constant, but which is not determined from the inner solution. This gives rise to a large temperature on the plate of $O(\sigma^{-2/5})$. This inner region drives a flow in a much thicker inviscid outer region, of thickness of $O(\sigma^{-2/5})$. At the outer edge of this outer region the ambient conditions are attained, and it is the matching between the two regions which determines the plate temperature.

1. Introduction

It was recognised from the outset by the original workers in free convection, [1, 2, 3], that the Prandtl number plays an important role in determining the nature of the flow and heat transfer in the boundary layer. Consequently there have been several studies in which this effect has been analysed in detail with asymptotic expansions being derived for both small and large values of the Prandtl number. Lefevre [4] was the first to write down the leading terms of the inner and outer expansions for both small and large Prandtl numbers, while at the same time a solution for large Prandtl number was obtained by Stewartson and Jones [5], who considered the free convection on an isothermal vertical plate. They showed that, in this limit, the boundary layer divided into two regions, a thin inner region in which the temperature decreased from its value on the plate to that of the ambient fluid and a much thicker outer region which was driven by the flow in the inner region and in which the fluid was at its ambient temperature. This work was later extended by Kuiken [6] and Eshghy [7]. The case when a uniform surface heat flux boundary condition is applied on the plate has been treated by Roy [8].

At the other end of the Prandtl number range, Kuiken [9] derived a solution for the free-convection boundary layer on an isothermal vertical plate which was valid for small Prandtl numbers. Here the boundary layer again divides into two regions, but now the inner region is, to leading order, isothermal (at the same temperature as the plate), with a thicker outer region which is effectively inviscid and at the outer edge of which the ambient conditions are attained.

It is the purpose of this paper to complete the discussion on the small Prandtl number solutions for a vertical plate by considering the case when a prescribed surface heat flux rather than a given temperature variation is prescribed on the plate. (This situation, as far as we are aware, has not been treated previously.) The case we consider in detail is when this applied heat flux is proportion to x^{λ}, in which case, [10], the governing equations can be reduced to similarity form. The solution is similar to that found by Kuiken [9] for an isothermal plate in that the boundary layer divides up into two regions, with the inner one again being isothermal, but here we have to determine this temperature by matching with the outer inviscid region. Also, we find that the scalings for the two regions are different to the isothermal plate case.

2. Equations

The basic equations are set out in Merkin [10] and need not be repeated fully here. Taking the condition on the temperature $(\partial T/\partial y)_{y=0} = -x^{\lambda}$, where T is the temperature and x and y are co-ordinates measuring distance along and perpendicular to the plate, respectively (all variables are non-dimensional), the boundary-layer equations for free convection on a vertical plate can be reduced to the ordinary differential equations

$$f''' + \theta + \left(\frac{4+\lambda}{5}\right) ff'' - \left(\frac{3+2\lambda}{5}\right) f'^2 = 0 \,, \tag{1a}$$

$$\frac{1}{\sigma}\theta'' + \left(\frac{4+\lambda}{5}\right) f\theta' - \left(\frac{1+4\lambda}{5}\right) f'\theta = 0 \,, \tag{1b}$$

where primes denote differentiation with respect to the independent variable η. The equations for the uniform heat flux case ($\lambda = 0$) were derived originally by Sparrow and Gregg [2]. To obtain equations (1) we have written

$$\psi = x^{(4+\lambda)/5} f(\eta) \,, \qquad T = x^{(1+4\lambda)/5}\theta(\eta) \,, \qquad \eta = yx^{(\lambda-1)/5} \tag{2}$$

where ψ is the stream function.

The boundary conditions to be applied are

$$f(0) = 0 \,, \qquad f'(0) = 0 \,, \qquad \theta'(0) = -1 \,, \quad f' \to 0, \ \theta \to 0, \text{ as } \eta \to \infty \,. \tag{3}$$

Here σ is the Prandtl number and we are looking for a solution of equations (1) subject to boundary conditions (3) valid for $\sigma \ll 1$.

It has been shown by Merkin [10] that equations (1) and (3) possess a solution only if $\lambda > -1$ and we will assume that this condition on λ holds throughout in the present work.

3. Solution

Equations (1) are similar to those considered by Kuiken [9] for the isothermal plate problem (though the boundary condition on θ on $\eta = 0$ is different) so it is natural as a first attempt at obtaining a solution for $\sigma \ll 1$ to follow Kuiken and leave equations (1) unscaled. Then equation (1b) gives $\theta'' = 0$, and so using (3) $\theta = a - \eta$ for some constant a. When this is then substituted into equation (1a) and a solution for large η sought we find that this has to be of the form $f \sim A\eta^{3/2} + \cdots$ for some constant A. However, when this functional form for f is tried in equation (1a) we get $A^2 = -\frac{4}{3}(1+\lambda)^{-1} < 0$ (since $1 + \lambda > 0$) which is unacceptable. This leads to the conclusion that the inner region must also be scaled.

A more detailed consideration of equations (1) shows that this inner region has a thickness of $O(\sigma^{1/10})$ and so is thin relative to the isothermal plate case (where the inner region is of

O(1)). This then suggests putting

$$f = \sigma^{-1/10} F(\zeta)\,, \qquad \theta = \sigma^{-2/5} H(\zeta)\,, \qquad \zeta = \sigma^{-1/10}\eta\,. \tag{4}$$

Using (4), equations (1) become

$$F''' + H + \left(\frac{4+\lambda}{5}\right) FF'' - \left(\frac{3+2\lambda}{5}\right) F'^2 = 0\,, \tag{5a}$$

$$H'' + \sigma\left[\left(\frac{4+\lambda}{5}\right) FH' - \left(\frac{1+4\lambda}{5}\right) F'H\right] = 0\,, \tag{5b}$$

subject to the boundary conditions on $\zeta = 0$ that

$$F(0) = 0\,, \qquad F'(0) = 0\,, \qquad H'(0) = -\sigma^{1/2}\,. \tag{6}$$

The outer boundary conditions are relaxed at this stage (primes now denote differentiation with respect to the scaled inner variable ζ).

Boundary conditions (6) suggest looking for a solution by expanding:

$$F = F_0 + \sigma^{1/2} F_1 + \cdots\,, \qquad H = H_0 + \sigma^{1/2} H_1 + \cdots\,. \tag{7}$$

Equation (5b) then gives $H_0'' = 0$ and since from (6) $H_0'(0) = 0$, we get

$$H_0 = a_0 \tag{8}$$

where a_0 is a constant which will be determined from matching with the outer region. We note that to leading order the inner region is isothermal but that its temperature is large, of $O(\sigma^{-2/5})$, and is not determined from the solution in the inner region but will be fixed by matching with the outer region. In this respect it is different from the prescribed temperature case.

Using (8), equation (5a) becomes

$$F_0''' + a_0 + \left(\frac{4+\lambda}{5}\right) F_0 F_0'' - \left(\frac{3+2\lambda}{5}\right) F_0'^2 = 0\,. \tag{9}$$

To solve this equation we scale the unknown constant a_0 out of equation (9) by writing

$$F_0 = \left(\frac{5}{3+2\lambda}\right)^{3/4} a_0^{1/4} \bar{F}_0\,, \qquad \bar{\zeta} = \left(\frac{3+2\lambda}{5}\right)^{1/4} a_0^{1/4} \zeta \tag{10}$$

(we are assuming that $a_0 > 0$, which we find is the case from the matching and is the only physically realistic possibility). Equation (9) then becomes

$$\bar{F}_0''' + \left(\frac{4+\lambda}{3+2\lambda}\right) \bar{F}_0 \bar{F}_0'' + 1 - \bar{F}_0'^2 = 0 \tag{11}$$

(primes denote differentiation with respect to $\bar{\zeta}$). From (11) we can readily see that the outer condition on $\bar{F}_0$ must be

$$\bar{F}_0 \sim \bar{\zeta} + \bar{b}_0 \tag{12}$$

for some constant $\bar{b}_0$.

Using condition (12) equation (11) can then be integrated numerically. We find that, for $\lambda = 0$ (uniform plate heat flux) $\bar{F}_0''(0) = 1.25913$ and $b_0 = -0.61782$.

At $O(\sigma^{1/2})$, equation (5b) gives $H_1'' = 0$ and then since $H_1'(0) = -1$ we obtain

$$H_1 = -\zeta + a_1 \tag{13}$$

for some further constant a_1. We then use (13) in equation (5a) to obtain an equation for F_1, which is then scaled using (10) for F_0 and ζ and by putting $F_1 = 5(3+2\lambda)^{-1} a_0^{-1} \bar{F}_1$. This then gives the equation for $\bar{F}_1$ as

$$F_1''' + \bar{a}_1 - \bar{\zeta} + \left(\frac{4+\lambda}{3+2\lambda}\right)(\bar{F}_0 \bar{F}_1'' + \bar{F}_0'' \bar{F}_1) - 2\bar{F}_0' \bar{F}_1' = 0 \tag{14}$$

where $\bar{a}_1 = a_1 a_0^{1/4} (3+2\lambda)^{1/4} 5^{-1/4}$. The boundary conditions are $\bar{F}_1(0) = 0$, $\bar{F}_1'(0) = 0$ and, from equation (14), that

$$F_1 \sim -\left(\frac{3+2\lambda}{2(2+3\lambda)}\right)\bar{\zeta}^2 + \frac{1}{2}\left\{\bar{a}_1 - \left(\frac{4+\lambda}{3\lambda+2}\right)\bar{b}_0\right\}\bar{\zeta} + \bar{b}_1 \tag{15}$$

as $\bar{\zeta} \to \infty$, where $\bar{b}_1$ is another constant.

We notice that equation (14) involves the as yet unknown constant $\bar{a}_1$ and so this equation cannot be solved at this stage. The unknowns a_0 and $\bar{a}_1$ are determined from matching with the solution in the outer region, which is what we consider next. Before doing so, however, we note that at $O(\sigma)$ we obtain $H_2'' = \frac{1}{5}(1+4\lambda)\, a_0 F_0'$, so that for $\zeta \gg 1$:

$$H_2 \sim \left(\frac{1+4\lambda}{5}\right)\left(\frac{5a_0}{3+2\lambda}\right)^{1/2} a_0 \frac{\zeta^2}{2} + a_2\zeta + c_2 \tag{16}$$

where a_2 and c_2 are further constants. The first and second terms in (16) contribute to the O(1) and $O(\sigma^{1/2})$ solutions respectively in the outer region.

To obtain the scalings for the outer region we first note that in the inner region θ is of $O(\sigma^{-2/5})$ and that f is of $O(\sigma^{-1/5}\eta)$ at the outer edge of the region. Using this and the requirement that in the outer region all the terms in equation (1b) should balance we arrive at the scalings for the outer region as

$$f = \sigma^{-3/5}\phi(Y), \qquad \theta = \sigma^{-2/5}h(Y), \qquad Y = \sigma^{2/5}\eta\,. \tag{17}$$

From (17) we see that the thickness of the outer region is of $O(\sigma^{-2/5})$ compared with a slightly larger thickness of $O(\sigma^{-1/2})$ for the prescribed temperature case, [9].

Using equation (17), equations (1) become

$$h + \left(\frac{4+\lambda}{5}\right)\phi\phi'' - \left(\frac{3+2\lambda}{5}\right)\phi'^2 + \sigma\phi''' = 0\,, \tag{18a}$$

$$h'' + \left(\frac{4+\lambda}{5}\right)\phi h' - \left(\frac{1+4\lambda}{5}\right)\phi' h = 0 \tag{18b}$$

(where primes now denote differentiation with respect to the outer variable Y). The boundary conditions for large Y are that

$$\phi' \to 0, \qquad h \to 0. \tag{19}$$

The behaviour of the solution for small Y is obtained from matching with the solution in the inner region (as $\zeta \to \infty$). Consider θ first. From (8), (13) and (16) and using the result that $\zeta = \sigma^{-1/2} Y$, we have

$$h = a_0 - Y + \left(\frac{1+4\lambda}{5}\right)\left(\frac{5a_0}{3+2\lambda}\right)^{1/2} \frac{a_0}{2} Y^2 + \cdots + \sigma^{1/2}(a_1 + a_2 Y + \cdots) + \cdots \tag{20a}$$

for $Y \ll 1$. To find the behaviour of ϕ for $Y \ll 1$ we use the asymptotic forms of (12) and (15) for $\bar{F}_0$ and $\bar{F}_1$ and then the scaling (10) to obtain

$$\begin{aligned}\phi = {} & \left(\frac{5a_0}{3+2\lambda}\right)^{1/2} Y - \frac{5}{2(2+3\lambda)}\left(\frac{3+2\lambda}{5a_0}\right)^{1/2} Y^2 + \cdots \\ & + \sigma^{1/2}\left\{ b_0 + \frac{1}{2}\left[a_1 - \left(\frac{4+\lambda}{2+3\lambda}\right)\left(\frac{3+2\lambda}{5a_0}\right)^{1/2} b_0\right]\left(\frac{5}{3+2\lambda}\right)^{1/2} \frac{Y}{a_0^{1/2}} + \cdots\right\} + \cdots\end{aligned} \tag{20b}$$

where

$$b_0 = \bar{b}_0 a_0^{1/4}\left(\frac{5}{3+2\lambda}\right)^{3/4}.$$

The form of boundary conditions (20a, b) suggests looking for a solution of equation (18) in the form

$$\phi = \phi_0 + \sigma^{1/2}\phi_1 + \cdots, \qquad h = h_0 + \sigma^{1/2} h_1 + \cdots. \tag{21}$$

The leading-order terms satisfy the equations

$$h_0 + \left(\frac{4+\lambda}{5}\right)\phi_0\phi_0'' - \left(\frac{3+2\lambda}{5}\right)\phi_0'^2 = 0, \tag{22a}$$

$$h_0'' + \left(\frac{4+\lambda}{5}\right)\phi_0 h_0' - \left(\frac{1+4\lambda}{5}\right)\phi_0' h_0 = 0 \tag{22b}$$

with, from the matching conditions (20), that

$$h_0 = a_0 - Y + \left(\frac{1+4\lambda}{5}\right)\left(\frac{5a_0}{3+2\lambda}\right)^{1/2} a_0 \frac{Y^2}{2} + \cdots, \tag{23a}$$

$$\phi_0 = \left(\frac{5a_0}{3+2\lambda}\right)^{1/2} Y - \frac{5}{2(2+3\lambda)}\left(\frac{3+2\lambda}{5a_0}\right)^{1/2} Y^2 + \cdots \tag{23b}$$

for $Y \ll 1$, and $\phi_0' \to 0$, $h_0 \to 0$ as $Y \to \infty$.

It would appear at first sight that the system given by (22) and (23) is over-constrained in that it is a fourth-order system with only three arbitrary constants (a_0 and two from the conditions for Y large). A further consideration of equations (22) reveals that the next term in (23b) should be of $O(Y^{5(2+\lambda)/(4+\lambda)})$ which does not appear from the expansion in the inner region in powers of $\sigma^{1/2}$, so that (23b) should be amended to

$$\phi_0 = \left(\frac{5a_0}{3+2\lambda}\right)^{1/2} Y - \frac{5}{2(2+3\lambda)}\left(\frac{3+2\lambda}{5a_0}\right)^{1/2} Y^2 + d_0 Y^{5(2+\lambda)/(4+\lambda)} + \cdots \tag{23c}$$

where d_0 is a further constant. This form for (23c) then in turn requires that the expansion (7) in the inner region be modified to include a term of $O(\sigma^{(3+2\lambda)/(4+\lambda)})$. A similar situation was found by Kuiken [9] for the prescribed temperature case.

The system given by equations (22) and (23a, c) can now be solved numerically, the solution determines the constants a_0 and d_0. A little care is needed in doing this as equation (22a) has a singularity at $Y=0$, and to overcome this the numerical integration was started at a small but non-zero value of Y, with the forms for ϕ_0 and h_0 as given by (23a, c) used to start the integration. The starting value of Y was successively reduced until the solution (and the values of a_0 and d_0) did not change to the required accuracy; a value of $Y=0.001$ was used for the results quoted. On performing these calculations, we found that (for $\lambda=0$) $a_0=1.31411$, $d_0=0.11672$ and $\phi(\infty)=1.37056$.

The equations for ϕ_1 and h_1 are

$$h_1 + \left(\frac{4+\lambda}{5}\right)(\phi_0\phi_1'' + \phi_0''\phi_1) - \frac{2(3+2\lambda)}{5}\,\phi_0'\phi_1' = 0\,, \tag{24a}$$

$$h_1'' + \left(\frac{4+\lambda}{5}\right)(\phi_0 h_1' + \phi_1 h_0') - \left(\frac{1+4\lambda}{5}\right)(\phi_0' h_1 + \phi_1' h_0) = 0\,, \tag{24b}$$

to be solved subject to the conditions $\phi_1' \to 0$, $h_1 \to 0$ as $Y \to \infty$ and, from (20), that

$$h_1 = a_1 + a_2 Y + \cdots\,, \tag{25a}$$

$$\phi_1 = b_0 + \frac{1}{2}\left[a_1 - \left(\frac{4+\lambda}{2+3\lambda}\right)\left(\frac{3+2\lambda}{5a_0}\right)^{1/2} b_0\right]\left(\frac{5}{3+2\lambda}\right)\frac{Y}{a_0^{1/2}} + \cdots\,. \tag{25b}$$

As for equations (22) the numerical investigation had to be started at a small non-zero value of Y. To get a solution it was found to be necessary to expand the solution of equations (24) for small Y to include the terms of $O(Y^2)$ in (25a) and of $O(Y^{2(3+2\lambda)/(4+\lambda)})$ in (25b). The solution at this stage appeared to be sensitive to very small changes in the leading order solution and so a solution could not be obtained to the same degree of accuracy as was possible for equations (22). We found that, for $\lambda=0$, $a_1=0.273$, $a_2=0.109$ and $\phi_1(\infty)=-0.100$.

Having determined a_0 and a_1 the value of $\bar{a}_1$ can be found. For $\lambda=0$ we get $\bar{a}_1=0.257$. This value can then be used to complete the solution of equations (14), the equations for the first perturbation in the inner region. We found that $\bar{F}_1''(0)=-1.099$ from which we get finally that $(d^2F_0/d\zeta^2)_0=1.75690$ and $(d^2F_1/d\zeta^2)_0=-1.237$.

The expansion in the inner and outer regions can be continued to higher-order terms, as seen above the next terms will be of $O(\sigma^{3/4})$. The matching between the two regions then determines the further constants which arise. This is not pursued further here.

4. Results

We have shown that for small values of the Prandtl number σ, the solution of equations (1) gives rise to a large plate temperature, of $O(\sigma^{-2/5})$ and a skin friction (related to $(d^2f/d\eta^2)_{\eta=0}$) of $O(\sigma^{-3/10})$. In particular we have found that, for the constant heat flux case $(\lambda = 0)$,

$$\theta(0) = \sigma^{-2/5}(1.31411 + 0.257\sigma^{1/2} + \cdots)\,, \tag{26a}$$

$$f''(0) = \sigma^{-3/10}(1.75690 - 1.237\sigma^{1/2} + \cdots)\,. \tag{26b}$$

To check on the range of applicability of (26), the original equations (1) were solved for small values of the Prandtl number using a standard numerical matching procedure for solving two-point boundary value problems. As σ was decreased this became increasingly more difficult to do. Because of (17) it was necessary to apply the outer boundary condition at successively larger values of η, for $\sigma = 0.0002$ (the lowest value of σ for which solutions were obtained) a value of $\eta = 400$ was required. Also, since values of $f''(0)$ and $\theta(0)$ became increasingly larger as σ was reduced, and the behaviour at large η appeared sensitive to small changes in estimates for these, requiring accurate initial estimates for these quantities, σ could be decreased only in very small increments.

To compare these numerical results with (26), graphs of $\theta(0)$ and $f''(0)$ obtained from expansions (26) (shown by the broken lines) and from the full solution of equations (1) are shown in Figs. 1 and 2, respectively. These figures clearly show that the asymptotic

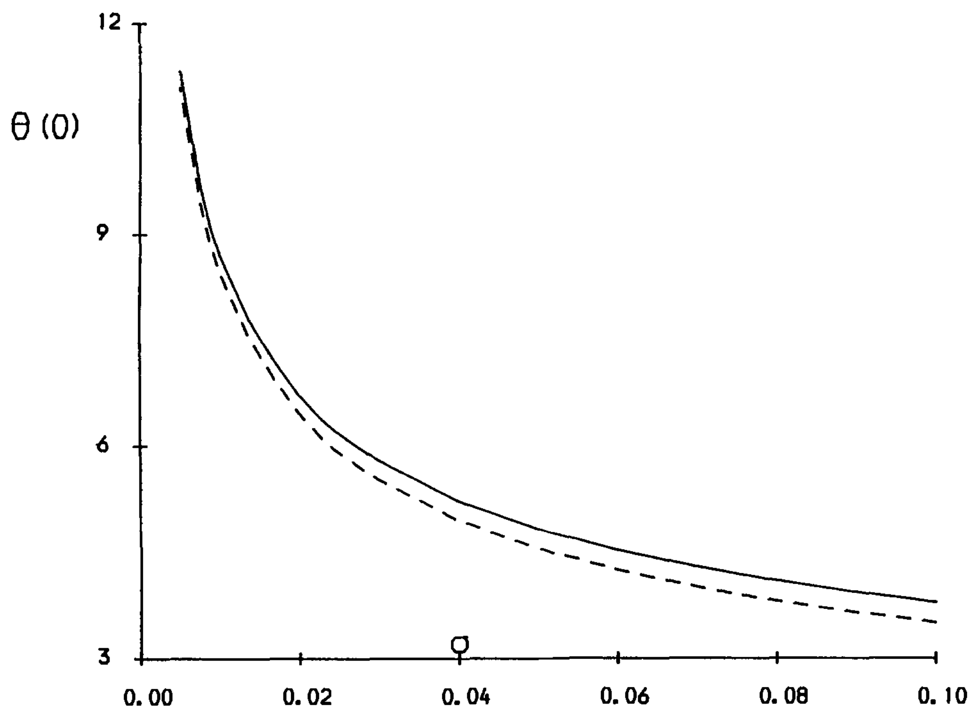

Fig. 1. Graphs of $\theta(0)$ obtained from the numerical solution of equations (1) (shown by the full line) and from series (26a) (shown by the broken line).

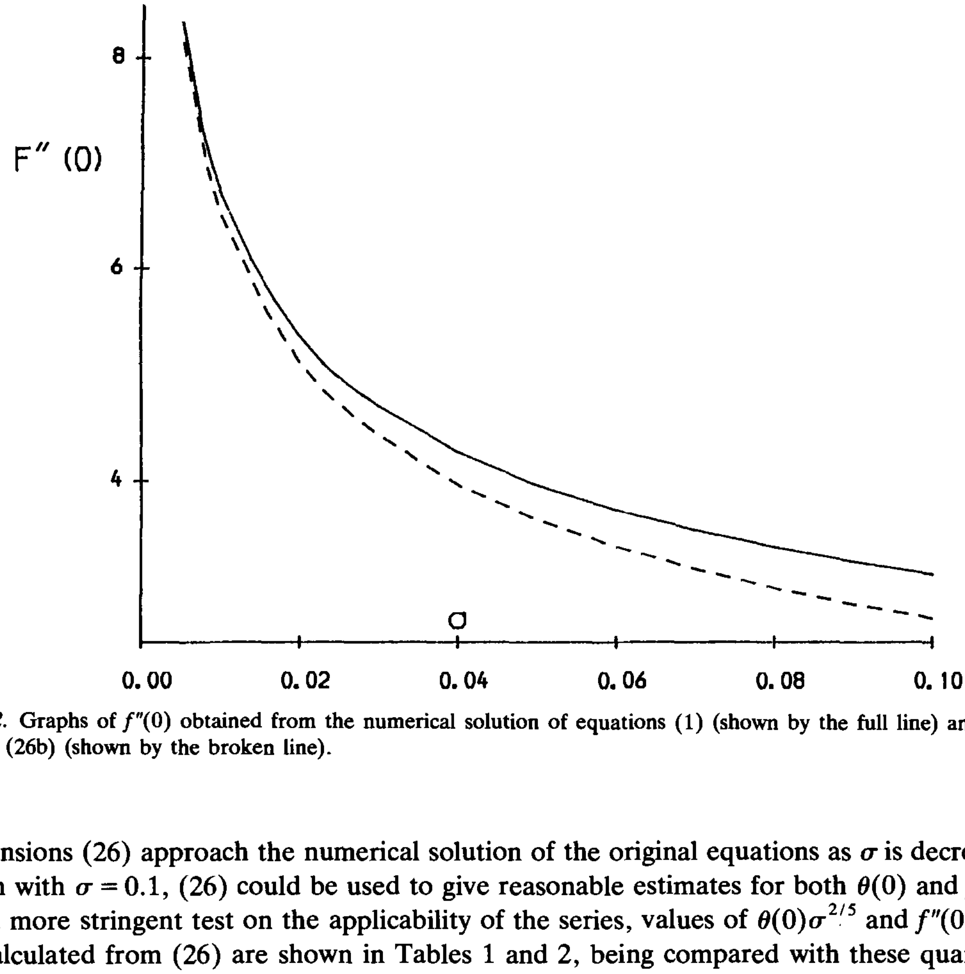

Fig. 2. Graphs of $f''(0)$ obtained from the numerical solution of equations (1) (shown by the full line) and from series (26b) (shown by the broken line).

expansions (26) approach the numerical solution of the original equations as σ is decreased. Even with $\sigma = 0.1$, (26) could be used to give reasonable estimates for both $\theta(0)$ and $f''(0)$. As a more stringent test on the applicability of the series, values of $\theta(0)\sigma^{2/5}$ and $f''(0)\sigma^{3/10}$ as calculated from (26) are shown in Tables 1 and 2, being compared with these quantities calculated from the numerical solutions of equations (1). Again we can see that there is very good agreement between the two for small σ.

The results given in Tables 1 and 2 (and Figs. 1 and 2) are derived from the solution in the inner region, and as a further check on the analysis, we consider the values of $f(\infty)$, as this is a quantity derived from the solution in the outer region. From (17) $f(\infty)$ is of $O(\sigma^{-3/5})$ for

Table 1. Values of $\theta(0)\sigma^{2/5}$ obtained from series (26a) and from the numerical solution of equations (1)

σ	Series	Numerical solution
0.0050	1.3323	1.3599
0.0025	1.3270	1.3466
0.0020	1.3256	1.3432
0.0015	1.3241	1.3393
0.0010	1.3222	1.3347
0.0008	1.3214	1.3326
0.0006	1.3204	1.3301
0.0004	1.3193	1.3272
0.0003	1.3186	1.3254
0.0002	1.3177	1.3234

Table 2. Values of $f''(0)\sigma^{3/10}$ obtained from series (26b) and from the numerical solution of equations (1)

σ	Series	Numerical solution
0.0050	1.6694	1.7025
0.0025	1.6951	1.7172
0.0020	1.7016	1.7210
0.0015	1.7090	1.7254
0.0010	1.7178	1.7308
0.0008	1.7219	1.7333
0.0006	1.7266	1.7362
0.0004	1.7322	1.7397
0.0003	1.7355	1.7418
0.0002	1.7394	1.7444

Table 3. Values of $f(\infty)\sigma^{3/5}$ obtained from series (27) and from the numerical solution of equations (1)

σ	Series	Numerical solution
0.0050	1.3635	1.3716
0.0025	1.3656	1.3711
0.0020	1.3661	1.3710
0.0015	1.3667	1.3709
0.0010	1.3674	1.3707
0.0008	1.3677	1.3708
0.0006	1.3681	1.3707
0.0004	1.3686	1.3707
0.0003	1.3690	1.3706
0.0002	1.3691	1.3706

for $\sigma \ll 1$, and for $\lambda = 0$ we found that

$$f(\infty) = \sigma^{-3/5}(1.37056 - 0.100\sigma^{1/2} + \cdots)\,. \tag{27}$$

Values of $\sigma^{3/5}f(\infty)$ calculated from (27) are given in Table 3 and compared with values obtained from the numerical solution of equations (1). Here again there is good agreement between the series and the numerical results. The slight oscillation in the last decimal place quoted in the values of $\sigma^{3/5}f(\infty)$ calculated from the numerical solutions suggests that the coefficient of the next term in the expansion, of $O(\sigma^{3/4})$ for $\lambda = 0$, makes a contribution comparable with the $O(\sigma^{1/2})$ term for the values of σ quoted in Table 3.

5. Conclusion

We have considered the behaviour of the solution of the equations for the free-convection boundary-layer flow on a vertical plate with a prescribed power-law heating for small values of the Prandtl number σ. We have shown that the boundary layer divides up into two regions, there is a thin inner region of thickness $O(\sigma^{1/10})$ in which the temperature is constant to leading order. This generates a large temperature on the plate, of $O(\sigma^{-2/5})$. This inner region sets up a flow in a much thicker inviscid outer region of thickness of $O(\sigma^{-2/5})$. At the outer edge of this outer region the ambient conditions are attained and it is the matching between the two regions that determines the plate temperature. A similar situation was found by Kuiken and Rotem [11] in their solution of the buoyant plume for large Prandtl numbers.

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